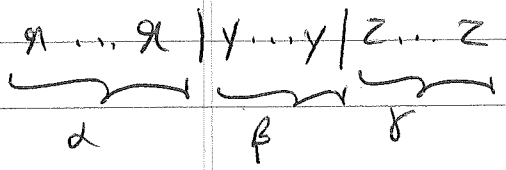


(1) Problem 4.3, Jackson.

(2) Problem 4.6, Jackson.

(1) Counting the number of $R_{\alpha\beta\gamma}^{(l)}$ is just counting the number of

ways to fill three boxes with two partitions such that $\alpha + \beta + \gamma = l$:



This number is given by:

$$1 + 2 + \dots + (l+1) = \frac{(l+1)(l+2)}{2} \quad *$$

For even l , the ^{total} number of spherical multipole moments is:

$$\sum_{l'=0, 2, \dots, l} (2l'+1) = \sum_{l'=0, 2, \dots, l} 2l' + (l+1) = 4 \sum_{\frac{l'}{2}=0}^{\frac{l}{2}} (\frac{l'}{2}) + (l+1) = \frac{4}{2} \frac{l}{2} (\frac{l}{2} + 1) + (l+1)$$

$$= (\frac{l}{2} + 1)(l+1) = \frac{(l+1)(l+2)}{2} \quad \leftarrow \text{Same as } *$$

For odd l , the total number follows:

$$\sum_{l'=1, 3, \dots, l} (2l'+1) = \sum_{n=1}^{\frac{l+1}{2}} (4n-1) = 4 \sum_{n=1}^{\frac{l+1}{2}} n - \frac{l+1}{2} = 2(\frac{l+1}{2})(\frac{l+3}{2}) - \frac{l+1}{2}$$

$$= \frac{(l+1)}{2} (l+3-1) = \frac{(l+1)(l+2)}{2} \quad \leftarrow \text{Same as } *$$

(2) (a) $W = -\frac{1}{6} \partial_i E_j(\cdot) Q_{ij}$

For a cylindrically symmetric field, we have $\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y}$. But:

$\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \Rightarrow \frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = -\frac{1}{2} \frac{\partial E_z}{\partial z}$

Also, note that $Q_{xy} = Q_{yz} = Q_{zx} = 0$ in this case. Since $Q_{xx} + Q_{yy} + Q_{zz} = 0$,

we have $Q_{xx} = Q_{yy} = -\frac{1}{2} Q_{zz}$.

symmetry of distribution

As a result, we have:

$W = -\frac{1}{6} (Q_{xx} \frac{\partial E_x}{\partial x} + Q_{yy} \frac{\partial E_y}{\partial y} + Q_{zz} \frac{\partial E_z}{\partial z}) = -\frac{1}{6} (\frac{1}{2} eQ \frac{\partial E_z}{\partial z} + eQ \frac{\partial E_z}{\partial z}) = -\frac{1}{4} eQ (\frac{\partial E_z}{\partial z})$

(b) This is a simple numerical calculation.

(c) $Q_{zz} = \int (3z^2 - r^2) \rho(\vec{x}) d\tau = \frac{Ze}{(\frac{4}{3}\pi a^2 b)} \int_{-b}^b dz \int_0^{a\sqrt{1-\frac{z^2}{b^2}}} \int_0^{2\pi} (2z^2 - \rho^2) \rho d\rho d\phi$

Here we have used:

$\rho(\vec{x}) = \frac{Ze}{\frac{4}{3}\pi a^2 b}$ (Volume of the spheroid $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$ is $\frac{4\pi}{3} a^2 b$)

Then:

$$Q_{zz} = \frac{3eZ}{4\pi a^2 b} \int_{-b}^b dz \left[\pi a^2 \left(1 - \frac{z^2}{b^2}\right) \cdot 2z^2 - 2\pi \int_0^{a\sqrt{1-\frac{z^2}{b^2}}} \rho^3 d\rho \right] =$$

$$\frac{3eZ}{4\pi a^2 b} 2\pi a^2 \left(\frac{2b^3}{3} - \frac{2b^3}{5} \right) - \frac{3Ze}{2a^2 b} \frac{a^4}{4} \int_{-b}^b dz \left(1 - \frac{z^2}{b^2}\right) =$$

$$\frac{2}{5} Ze b^2 - \frac{2}{5} Ze a^2 \Rightarrow Q_{zz} = \frac{2}{5} eZ (b^2 - a^2) \Rightarrow Q = \frac{2}{5} Z (b^2 - a^2)$$

Using $R = \frac{b+a}{2}$, we find:

$$\frac{b-a}{R} = \frac{5Q}{4R^2 Z} = \frac{5 \times 2.5 \times 10^{-28}}{4 \times 49 \times 10^{-30} \times 63} \approx 0.1$$