PHYC 511
Spring 2018

Problem Session 4 02/09/2018

(1) Problem 4.3, Jackson.

(2) Problem 4.6, Jackson.

(1) Counting the number of $Q^{(\ell)}_{\alpha\beta\gamma}$ is just counting the number of ways to fill three boxes with two partitions such that $\alpha+\beta+\gamma=\ell$:

$$\underbrace{x\ldots x}_{\alpha}\,|\,\underbrace{y\ldots y}_{\beta}\,|\,\underbrace{z\ldots z}_{\gamma}$$

This number is given by:

$$\underset{\substack{\uparrow\\ \alpha=\ell}}{1}+\underset{\substack{\uparrow\\ \alpha=\ell-}}{2}+\cdots+\underset{\substack{\uparrow\\ \alpha=0}}{(\ell+1)}=\boxed{\frac{(\ell+1)(\ell+2)}{2}} \quad *$$

For even ℓ, the total number of spherical multipole moments is:

$$\sum_{\ell'=0,2,\ldots,\ell}(2\ell'+1)=\sum_{\ell'=0,2,\ldots,\ell}2\ell'+\left(\frac{\ell}{2}+1\right)=4\sum_{\frac{\ell'}{2}=0}^{\frac{\ell}{2}}\left(\frac{\ell'}{2}\right)+\left(\frac{\ell}{2}+1\right)=\frac{4}{2}\,\frac{\ell}{2}\left(\frac{\ell}{2}+1\right)+\left(\frac{\ell}{2}+1\right)$$

$$=\left(\frac{\ell}{2}+1\right)(\ell+1)=\boxed{\frac{(\ell+1)(\ell+2)}{2}} \leftarrow \text{Same as } *$$

For odd ℓ, the total number follows:

$$\sum_{\ell'=1,3,\ldots,\ell}(2\ell'+1)=\sum_{n=1}^{\frac{\ell+1}{2}}(4n-1)=4\sum_{n=1}^{\frac{\ell+1}{2}}n-\frac{\ell+1}{2}=2\left(\frac{\ell+1}{2}\right)\left(\frac{\ell+3}{2}\right)-\frac{\ell+1}{2}$$

$$=\frac{(\ell+1)}{2}(\ell+3-1)=\boxed{\frac{(\ell+1)(\ell+2)}{2}} \longleftarrow \text{Same as } *$$

(2) (a) $W = -\frac{1}{6}\, \partial_i E_j(0)\, Q_{ij}$

For a cylindrically symmetric field, we have $\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y}$. But:

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \Rightarrow \boxed{\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = -\frac{1}{2}\frac{\partial E_z}{\partial z}}$$

Also, note that $Q_{xy} = Q_{yz} = Q_{zx} = 0$ in this case. Since $Q_{xx} + Q_{yy} + Q_{zz} = 0$,

we have $Q_{xx} \underset{\uparrow}{=} Q_{yy} = -\frac{1}{2} Q_{zz}$.

symmetry of distribution

As a result, we have:

$$W = -\frac{1}{6}\left(Q_{xx}\frac{\partial E_x}{\partial x} + Q_{yy}\frac{\partial E_y}{\partial y} + Q_{zz}\frac{\partial E_z}{\partial z}\right)_0 = -\frac{1}{6}\left(\frac{1}{2} eQ \frac{\partial E_z}{\partial z} + eQ\frac{\partial E_z}{\partial z}\right)_0 = \boxed{-\frac{1}{4} eQ\left(\frac{\partial E_z}{\partial z}\right)_0}$$

(b) This is a simple numerical calculation.

(c) $$Q_{zz} = \int (3z^2 - r^2)\, \rho(\vec{x})\, d\tau = \frac{Ze}{\left(\frac{4}{3}\pi a^2 b\right)} \int_{-b}^{b} dz \int_0^{a\sqrt{1-\frac{z^2}{b^2}}} \rho\,(2z^2 - \rho^2)\, d\rho \int_0^{2\pi} d\phi$$

Here we have used:

$$\rho(\vec{x}) = \frac{Ze}{\frac{4}{3}\pi a^2 b}$$ (Volume of the spheroid $\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1$ is $\frac{4\pi}{3}a^2 b$)

Then:

$$Q_{zz} = \frac{3eZ}{4\pi a^2 b} \int_{-b}^{b} dz \left[\pi a^2 \left(1 - \frac{z^2}{b^2}\right) \cdot 2z^2 - 2\pi \int_0^{a\sqrt{1-\frac{z^2}{b^2}}} \rho^3 \, d\rho \right] =$$

$$\frac{3eZ}{4\pi a^2 b} \, 2\pi a^2 \left(\frac{2b^3}{3} - \frac{2b^3}{5} \right) - \frac{3Ze}{2a^2 b} \, \frac{a^4}{4} \int_{-b}^{b} dz \left(1 - \frac{z^2}{b^2}\right) =$$

$$\frac{2}{5} Zeb^2 - \frac{2}{5} Zea^2 \Rightarrow Q_{zz} = \frac{2}{5} eZ (b^2 - a^2) \Rightarrow \boxed{Q = \frac{2}{5} Z (b^2 - a^2)}$$

Using $R = \frac{b+a}{2}$, we find:

$$\frac{b-a}{R} = \frac{5Q}{4R^2 Z} = \frac{5 \times 2.5 \times 10^{-28}}{4 \times 49 \times 10^{-30} \times 63} \boxed{\simeq 0.1}$$